Determination of vertical gradients from torsion balance measurements

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Abstract Vertical gravity gradients are measured in the field by gravimeters mainly for absolute gravity measurements but they are useful for gravity field determination as well. Using torsion balance measurements, however, it is possible to make a relative determination of vertical gravity gradients using an idea due to Haalck. Curvature values are differentiated and combined first to get horizontal variation of the vertical gravity gradient and then using a computation process similar to astronomic leveling vertical gravity gradient differences are yielded. Simulated vertical gradients of torsion balance type were compared with actual synthetic vertical gradients. A good agreement was found between the computed and analytically determined vertical gradients.

Keywords. vertical gravity gradients, Eötvös' torsion balance, synthetic gravitational field

1 Introduction

Torsion balance measurements provide horizontal gravity gradients T_{zx}, T_{zy} and curvature values $T_\Delta = T_{yy}-T_{xx}$, $2T_{xy}$, that is certain second derivatives of the disturbing potential T. One can find a clever concept in Haalck (1950) how to derive a complete picture of the local gravity field from these measurements. Following the above idea of Haalck in this paper we propose a simple method to determine relative changes of the vertical gravity gradient by using gravity field information provided for example by the Eötvös torsion balance.

In the first part of the paper we describe the computation procedure how to derive vertical gravity gradient variation along a traverse from measured gravity gradients and curvature values. In the next part a numerical example will be provided by using a synthetic gravity field model. Finally, conclusions and recommendations will be drawn for the practical application of the method.

2 The proposed method

Let us define the coordinate system by the *x, y,* and *z*-axis pointing to East, North and Up, respectively. Obviously, any other coordinate system can be used but it have to be consistent both for the coordinates of the points and for the measurements.

The difference of any torsion balance measurement $T \in \{T_{xx}-T_{yy}, T_{xy}, T_{xz}, T_{yz}\}$ can be expressed between the point P and points 1 or 2. With the notation of Figure 1

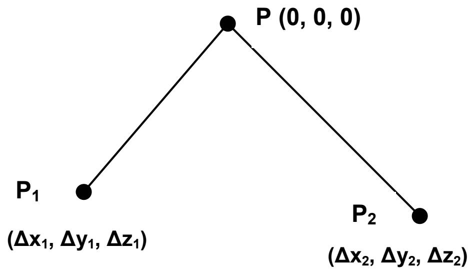

Fig. 1 Coordinates of torsion balance measurement sites P, P_1 and P_2.

$$\begin{bmatrix} \Delta T_1 \\ \Delta T_2 \end{bmatrix} = \begin{bmatrix} \Delta x_1 & \Delta y_1 & \Delta z_1 \\ \Delta x_2 & \Delta y_2 & \Delta z_2 \end{bmatrix} \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}. \qquad (1)$$

In the formula above T denotes an arbitrary kind of torsion balance measurement while T_x, T_y, T_z denotes unknown derivatives of T with respect to x, y and z. Since there are altogether 4 kinds of measurements, therefore theoretically the following 12 unknowns have to be determined between the 3 points of Figure 1. from the following $4 \times 2 = 8$ measurements

$$\begin{bmatrix} T_{\Delta x} & T_{xyx} & T_{xzx} & T_{yzx} \\ T_{\Delta y} & T_{xyy} & T_{xzy} & T_{yzy} \\ T_{\Delta z} & T_{xyz} & T_{xzz} & T_{yzz} \end{bmatrix}.$$

The determination of these 12 unknowns is possible only because additional 5 constraints can be set up between the unknowns due to symmetry reasons and also after taking suitable derivatives of the Laplace equation $T_{xx} + T_{yy} + T_{zz} = 0$. These constraints are the following:

$$
\begin{aligned}
T_{\Delta z} &= T_{xzx} - T_{yzy} \\
T_{xyz} &= T_{xzy} \\
T_{xzz} &= -T_{\Delta x} - 2T_{xyy} \\
T_{yzx} &= T_{xzy} \\
T_{yzz} &= T_{\Delta y} - 2T_{xyx}
\end{aligned}
\qquad (2a\text{-}e)
$$

Therefore it follows that for the remaining 7 unknowns 8 equations can be set up:

$$
\begin{bmatrix} \Delta T_{\Delta 1} \\ \Delta T_{\Delta 2} \\ \Delta T_{xy1} \\ \Delta T_{xy2} \\ \Delta T_{xz1} \\ \Delta T_{xz2} \\ \Delta T_{yz1} \\ \Delta T_{yz2} \end{bmatrix}
=
\begin{bmatrix}
\Delta x_1 & \Delta y_1 & 0 & 0 & \Delta z_1 & -\Delta z_1 & 0 \\
\Delta x_2 & \Delta y_2 & 0 & 0 & \Delta z_2 & -\Delta z_2 & 0 \\
0 & 0 & \Delta x_1 & \Delta y_1 & 0 & 0 & \Delta z_1 \\
0 & 0 & \Delta x_2 & \Delta y_2 & 0 & 0 & \Delta z_2 \\
-\Delta z_1 & 0 & 0 & -2\Delta z_1 & \Delta x_1 & 0 & \Delta y_1 \\
-\Delta z_2 & 0 & 0 & -2\Delta z_2 & \Delta x_2 & 0 & \Delta y_2 \\
0 & \Delta z_1 & -2\Delta z_1 & 0 & 0 & \Delta y_1 & \Delta x_1 \\
0 & \Delta z_2 & -2\Delta z_2 & 0 & 0 & \Delta y_2 & \Delta x_2
\end{bmatrix}
\begin{bmatrix} T_{\Delta x} \\ T_{\Delta y} \\ T_{xyx} \\ T_{xyy} \\ T_{xzx} \\ T_{yzy} \\ T_{xzy} \end{bmatrix}.
\qquad (3)
$$

This overdetermined linear system of equations can be solved for example by minimizing the sum square of differences on the left side derived from measurement and unknowns. Then the solution vector can be used to determine the following three derivatives of the vertical gravity gradient T_{zz} at P

$$
\begin{aligned}
T_{zzx} &= -T_{\Delta x} - 2T_{xyy} \\
T_{zzy} &= T_{\Delta y} - 2T_{xyx} \\
T_{zzz} &= -T_{xzx} - T_{yzy}
\end{aligned}
\quad . \qquad (4a\text{-}c)
$$

The above three equations again can easily be deduced from the Laplace equation by differentiation.

Extending the above to a traverse with more than 3 points, at all points the above 3 derivatives of T_{zz} can be determined, except at the endpoints. Finally starting with a known T_{zz} value and summing the differences

$$
[\Delta T_{zz}]_{i,i+1} = [\Delta x \quad \Delta y \quad \Delta z]_{i,i+1} \begin{bmatrix} T_{zzx} \\ T_{zzy} \\ T_{zzz} \end{bmatrix}_{i,i+1} \qquad (5)
$$

between points (i, i+1), the vertical gravity gradient T_{zz} can be calculated for each point of the traverse. At the midpoints the two set of 3 derivatives of T_{zz} can simply be averaged.

3 Numerical example

We can illustrate the above procedure by defining a simple 3D density model and computing all the necessary parameters of its gravitational field. This has the advantage that a self-consistent dataset of all the required gravity field parameters can be computed analytically. Moreover, by using such a synthetic model, the approximation errors of the computational procedure can easily be assessed.

Let us define a simple density source: we place a cube of 20 m size with density ρ = 2670 kg/m^3 at 20 m below the zero level (z = 0), i. e. its centre lies at the point with coordinates of (0, 0, -30) m. Next a traverse of 11 points is defined. The coordinates of this traverse line are shown in Table 1.

Table 1. Coordinates of the computation points of the traverse.

Point	x [m]	y [m]	z [m]
1.	-13.995	19.240	1.00
2.	-8.080	13.995	1.50
3.	0.915	13.415	2.50
4.	1.250	7.835	0.00
5.	-1.830	3.170	1.00
6.	7.165	2.590	2.50
7.	8.750	-5.155	4.00
8.	13.415	-8.236	5.50
9.	12.500	-11.651	6.50
10.	7.255	-17.566	6.00
11.	14.085	-19.396	4.50

The total length of the traverse is 69.12 m, hence the average distance between two points is 6.91 m and the maximum height difference is 6.5 m.

The second derivatives of the gravitational potential of this body (cf. Table 2) were computed using the formulas of a general polyhedral body published by Holstein (2003).

Table 2. Gravitational gradients computed from the density model. All units are Eötvös (1E = 10^{-9} s^{-2}).

Point	V_{xx} [E]	V_{xy} [E]	V_{xz} [E]	V_{yy} [E]	V_{yz} [E]	V_{zz} [E]
1.	-14.86	-12.51	-20.37	-6.74	28.13	21.61
2.	-27.08	-8.33	-19.20	-17.53	33.49	44.61
3.	-32.21	0.92	2.29	-18.93	33.91	51.15
4.	-46.11	1.30	5.22	-38.44	32.98	84.55
5.	-44.96	-0.74	-7.61	-44.14	13.19	89.10
6.	-32.74	1.75	22.92	-36.84	8.25	69.58
7.	-26.03	-3.21	21.90	-29.53	-12.86	55.56
8.	-15.80	-5.17	22.76	-21.01	-13.93	36.80
9.	-15.58	-5.70	18.16	-16.38	-16.91	31.95
10.	-19.06	-4.76	9.88	-9.54	-24.05	28.60
11.	-12.83	-8.83	15.86	-7.08	-21.90	19.91

Vertical gravity gradients were computed at each point with the procedure described in the previous section (the value of V_{zz} calculated at the first point was used to initialize the computation). The computational results and their differences with respect to the synthetic vertical gravity gradients (last column of Table 2.) are shown below in Table 3.

Table 3. Vertical gravity gradients computed from simulated torsion balance measurements and their differences with respect to their "true" analytic values at each point of the traverse.

Point	V_{zz} [E] (computed)	ΔV_{zz} [E] (computed-analytic)
1.	-13.99	19.24
2.	-8.08	13.99
3.	0.92	13.41
4.	1.25	7.83
5.	-1.83	3.17
6.	7.16	2.59
7.	8.75	-5.15
8.	13.41	-8.24
9.	12.50	-11.65
10.	7.25	-17.57
11.	14.08	-19.40

The standard deviation of differences is ± 4.68 E, which is about 20.5 % of the mean square value ± 22.79 E of the vertical gradients themselves. The computed and analytical ("true") values of the vertical gravity gradients are shown in Figure 2, referring to a common mean value for easier comparison.

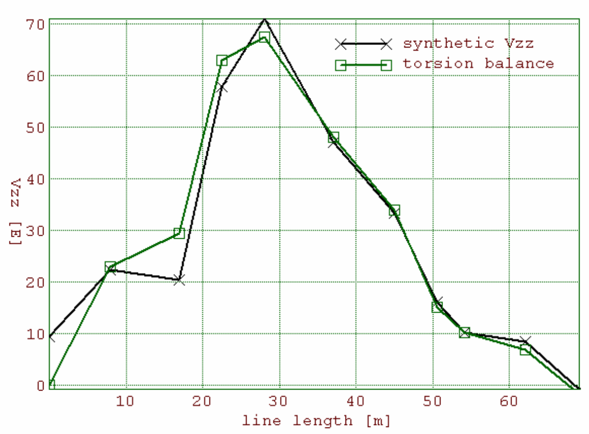

Fig. 2 Vertical gravity gradients V_{zz}, computed from simulated torsion balance measurements (denoted by □) and from the analytical model (denoted by ×). Horizontal axis is the line length in meter, vertical axis is V_{zz} in Eötvös.

If we reduce the length of the line by 50% but keep its origin, the standard deviation of differences also reduces to ± 1.29 E, which is only 12.9 % of the mean square value ± 10.05 E of the vertical gradients themselves.

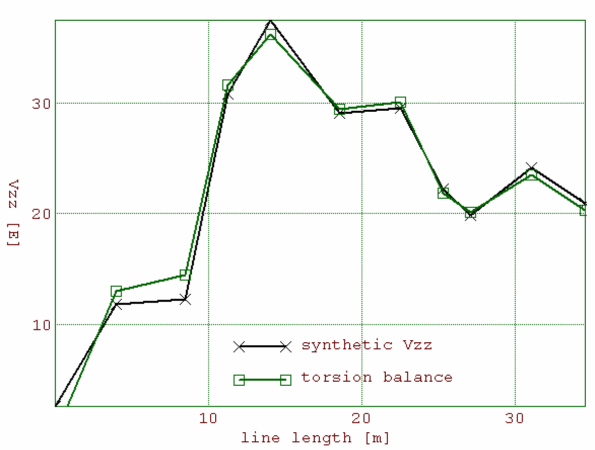

Fig. 3 Vertical gravity gradients V_{zz}, computed from simulated torsion balance measurements (denoted by □) and from the analytical model (denoted by ×). Same as Figure 2, but the line length is reduced by 50%.

This simple check shows (in agreement with our expectations) the substantial reduction of the linearization error of the computation by decreasing the distance of torsion balance stations. Of course the optimal distance of points also depends on the local structure of the gravity field and the cost of measurements.

The measurement errors of the torsion balance also have a substantial effect on the computed vertical gravity gradients. These errors have been modeled by adding random noise to the density model generated gradient and curvature measurements of the Eötvös torsion balance. It is evident from Table 4. that the standard deviation of computed V_{zz} values with respect to their error-free values increases more rapidly by increasing the noise level of curvature terms than that of the gradient terms. When the height differences of points is greater, however, the figures in Table 4 show that the errors of the measured horizontal gradients have greater impact on the error of computed V_{zz} values. Therefore we conclude that the geometry of computation points have a strong effect on the error of vertical gravity gradients.

Table 4. Standard deviation of differences of vertical gravity gradients due to simulated torsion balance gradient and curvature measurement errors. The maximum height difference of points is also varying. All units are Eötvös (1E = 10^{-9} s^{-2}).

$\sigma_{curv} \rightarrow$ $\sigma_{grad} \downarrow$	Δh_{max} [m]	± 1 E	± 2 E	± 3 E
± 1 E		± 5.8	± 11.5	± 17.3
± 2 E	1.3	± 6.1	± 11.5	± 17.3
± 3 E		± 6.3	± 11.8	± 17.9
± 1 E		± 4.7	± 8.9	± 13.2
± 2 E	6.5	± 6.0	± 9.8	± 13.6
± 3 E		± 7.5	± 10.6	± 14.6
± 1 E		± 7.0	± 9.0	± 11.9
± 2 E	32.5	± 12.3	± 13.1	± 15.4
± 3 E		± 18.2	± 18.9	± 20.4

4 Conclusions and recommendations

A procedure was presented to compute differences of vertical gravity gradients from torsion balance measurements. Our checks with a simple synthetic gravity field model have shown that the computation is feasible. The accuracy of the determined vertical gravity gradients depends on many factors, but the most serious are the measurement and discretization (linearization) errors. The main driving factors are linearization errors, which can be reduced by decreasing the distance between torsion balance measurement sites, depending on the structure of the gravity field, and point geometry. The larger the height variation of the points, the more accurate measurements of horizontal gravity gradients is needed. On the contrary, if the horizontal extent of the computation area is larger than the vertical one, it is recommended to increase the accuracy of curvature gravity gradients.

The accuracy of the relative vertical gravity gradient determination from torsion balance measurements, however, is expected to surpass the accuracy obtainable by gravimeters, which is about ± 30 E for two measurement series with 4 gravimeters (Csapó and Völgyesi, 2003).

The proposed procedure may have practical application in the future in those areas where torsion balance measurements exist. For example, test computations are in preparation on the network points of the Budapest microbase. This small network contains 14 torsion balance stations, and each station has vertical gradient value measured by LCR gravimeters. Sometimes we need a map of the vertical gravity gradients for some reason (for example for gravity field/geoid determination). For this purpose other methods of computation (e.g. gradient kriging, see Menz and Knospe, 2002) may be more feasible to extend the computation over scattered data points instead of along a traverse line.

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