LOCAL GEOID DETERMINATION BASED ON GRAVITY GRADIENTS

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There is a good possibility to interpolate a dense net of deflections of the vertical from $W_{yy} - W_{xx}$ and W_{xy} gravity gradients measured by torsion balance and applying astronomical levelling to compute geoid heights. A new practical computation of astronomical levelling is suggested.

Keywords: local geoid, astronomical levelling, deflection of the vertical, gravity gradients, torsion balance measurements

1. Introduction

Nowadays, based on the results of up-to-date geodetic measurements, contour line maps of the main geoid forms for the whole Earth are available. These global geoid forms do not contain the "fine structure" of the geoid, while we need it at least centimetres accuracy for practical geodetic purposes (e.g. determination of height above see level from GPS measurement). A dense net of deflections of the vertical, interpolated from gravity gradients measured by torsion balance gives us a good possibility to determine a detailed local geoid map.

2. Astronomical levelling

The basic principle of astronomical levelling gives us a definite mathematical relationship between geoid undulations and deflections of the vertical (Biró 1982). According to the notations of *Fig. 1* we get

$$dN = \vartheta\, ds$$

where ϑ is the *Pizzetti-type* deflection of the vertical in the azimuth α.

Between any points P_1 and P_k the geoid height change is

$$\Delta N_{1k} = \int_1^k \vartheta(s) ds \tag{1}$$

where $\vartheta(s)$ is the function of deflection of the vertical in the azimuth α_{1k}, and s is the distance between the two points. For using this line integral we would have to know the function of deflection of the verticals $\vartheta(s)$ between the points.

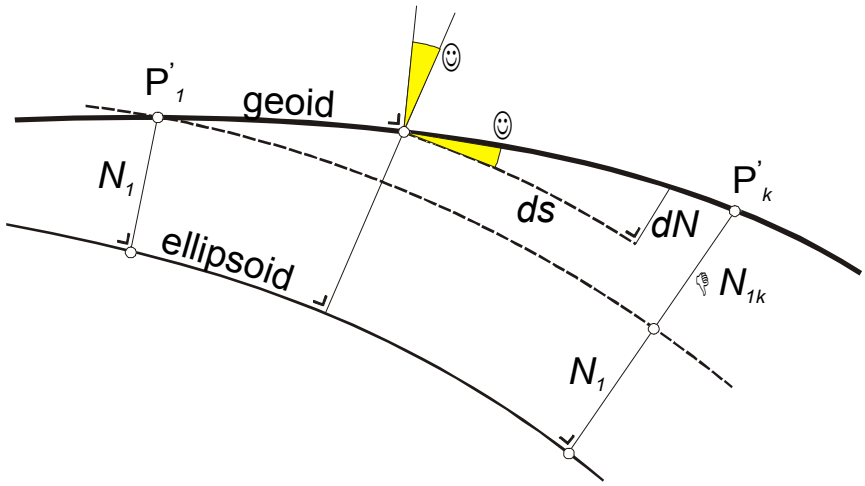

Fig. 1 Basic principle of astronomical levelling

If P_i and P_k are close together and $\vartheta(s)$ is a linear function between these points the integral (1) can be evaluated by a numerical integration:

$$\Delta N_{ik} = \frac{\vartheta_i + \vartheta_k}{2} s_{ik} \quad . \tag{2}$$

Unfortunately in practice we have a sparser net of astronomical stations, where deflections of the vertical are known and $\vartheta(s)$ can not be linear between the points P_i and P_k. So we need denser net of deflections of the vertical for astronomical levelling where $\vartheta(s)$ can be treated as linear function between neighbouring points. There is a very good possibility to make a denser net by interpolating deflections of the vertical between the astronomical points from gravity gradients where torsion balance measurements are available.

3. Interpolation of deflection of the vertical

A very simple relationship based on potential theory can be written for the changes of $\Delta\xi_{ik}$ and $\Delta\eta_{ik}$ between arbitrary points i and k of the deflection of the vertical components ξ and η as well as for gravity gradients W_{xx}, W_{yy} and W_{xy} measured by torsion balance:

$$\begin{aligned} &\Delta\xi_{ik} \sin\alpha_{ki} - \Delta\eta_{ik} \cos\alpha_{ki} = \\ &\frac{s_{ik}}{4g}\left\{\left[\left(W_\Delta - U_\Delta\right)_i + \left(W_\Delta - U_\Delta\right)_k\right]\sin 2\alpha_{ik} + \left[\left(W_{xy} - U_{xy}\right)_i + \left(W_{xy} - U_{xy}\right)_k\right]2\cos 2\alpha_{ik}\right\} \end{aligned} \tag{3}$$

where $W_\Delta = W_{yy} - W_{xx}$, $U_\Delta = U_{yy} - U_{xx}$, s_{ik} is the distance between points i and k, g is the average value of gravity between them, U_{xx}, U_{yy} and U_{xy} are gravity

gradients in the normal gravity field, whereas α_{ik} is the azimuth between the two points (Völgyesi 1993, 1995). Writing the left side of Eq. (3) in other form:

$$\xi_k \sin\alpha_{ki} - \xi_i \sin\alpha_{ki} - \eta_k \cos\alpha_{ki} + \eta_i \cos\alpha_{ki} =$$
$$\frac{s_{ik}}{4g}\left\{\left[\left(W_\Delta - U_\Delta\right)_i + \left(W_\Delta - U_\Delta\right)_k\right]\sin 2\alpha_{ik} + \left[\left(W_{xy} - U_{xy}\right)_i + \left(W_{xy} - U_{xy}\right)_k\right]2\cos 2\alpha_{ik}\right\} \tag{4}$$

The computation being fundamentally an integration, practically possible only by approximation, in deriving (3) or (4) it had to be assumed that the change of gravity gradients between points i and k, measurable by torsion balance, was linear – thus the equality sign in (3) or (4) is valid only for this case (Völgyesi 1993).

4. Classical computation of astronomical levelling

According to *Fig. 2*

$$\vartheta = \xi\cos\alpha + \eta\sin\alpha$$

and using ξ and η interpolated by (4) :

$$\Delta N_{ik} = \left(\frac{\xi_i + \xi_k}{2}\cos\alpha_{ik} + \frac{\eta_i + \eta_k}{2}\sin\alpha_{ik}\right)s_{ik} \tag{5}$$

where ΔN_{ik} is the geoid undulation difference between points P_i and P_k .

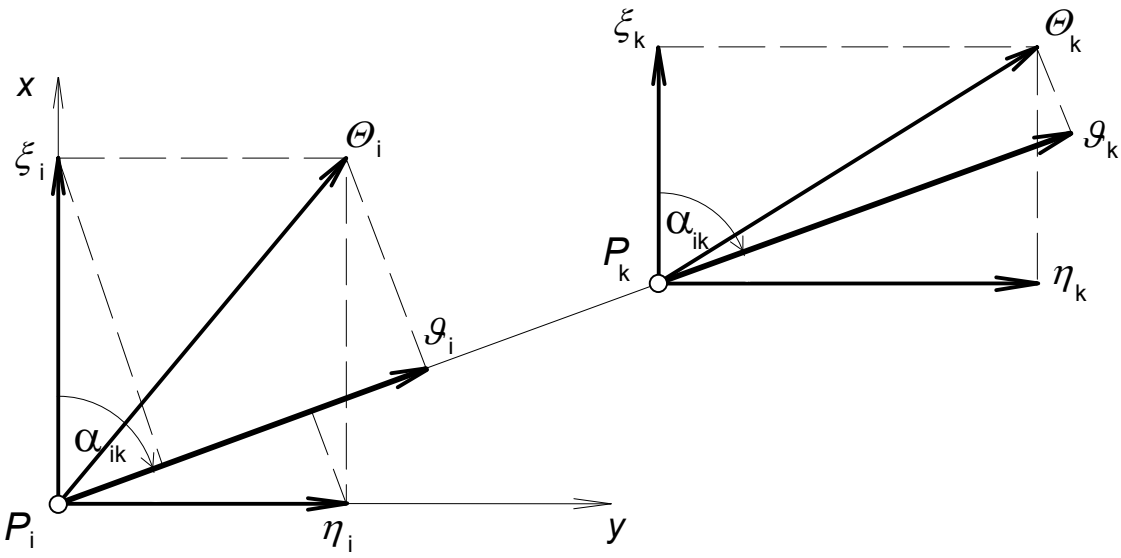

Fig. 2. Deflection of the vertical in an arbitrary azimuth α

Generally we want to compute the difference ΔN not only between two points but we need a detailed geoid map for a specific territory. According to the practical work up

till now if we had a denser net of deflections of the vertical, and we wanted to apply astronomical levelling for a larger territory, we would first interpolate deflection of the vertical components ξ and η from an arbitrary shaped network's points (in our case the torsion balance stations) to grid points of a square-shaped network. Then applying Eq. (5) we could compute differences of geoid heights, along grid points of the square-shaped network. In the *north-south* profiles $\alpha_{ik} = 0^{\circ}$ hence from (5) :

$$\Delta N_{ik}^{N-S} = \left(\frac{\xi_i + \xi_k}{2} \right) s_{ik}$$

and the *east-west* profiles $\alpha_{ik} = 90^{\circ}$ hence from (5) :

$$\Delta N_{ik}^{E-W} = \left(\frac{\eta_i + \eta_k}{2} \right) s_{ik} \quad .$$

After computing all the north-south and east-west profiles we'll get ΔN differences for each square sides. Going around each square of the square-shaped network the sum of ΔN differences for the four square sides must be zero. E.g. on *Fig.3* for the *j*th square:

$$\Delta N_{j}^{W \to E} + \Delta N_{j+1}^{N \to S} + \Delta N_{j+2}^{E \to W} + \Delta N_{j+3}^{S \to N} = 0 \ . \tag{6}$$

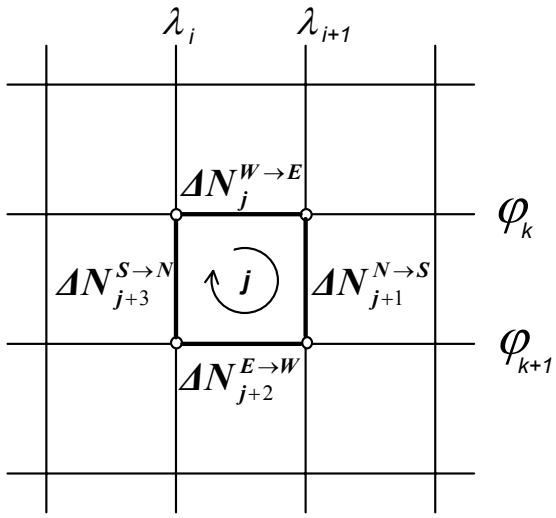

Fig. 3. Summing up ΔN differences for square sides

If the sum is not zero, these misclosures can be adjusted as if they were the misclosures of the well known geometrical levelling. Results of this computation will be the adjusted ΔN differences between the adjacent square-shaped grid points.

If an initial geoid height N_1 is given at an arbitrary point of the grid we can get the final geoid heights N_i at point i, summing up the corresponding ΔN differences.

Another solution using a least-squares surface fitting technique was made by (Vanicek P and Merry C L, 1993).

5. Practical computation of local geoid heights

There are two main problems of the classical computation of astronomical levelling. There will be a loss of accuracy of computed ξ, η values interpolating them from a network of points having arbitrary shape to grid points of a square-shaped network in one respect, and there are too many unknowns without reason considering geoid height changes ΔN between network points as unknowns instead of direct geoid heights N for each points as unknowns on the other hand (Völgyesi, 1998).

The first problem can be solved using the original torsion balance measurement points directly for the geoid computation instead of regular grid points. In this case we use a net of triangles instead of squares, and (5) gives the relationship between components of deflection of the vertical ξ, η and the geoid height change ΔN for each triangle sides in an arbitrary azimuth α.

The second difficulty may be overcome by considering N values of geoid heights directly as unknowns instead of differences ΔN for the same arbitrary network points. Accordingly, let us transform (5) by substituting:

$$\Delta N_{ik} = N_i - N_k$$

to

$$N_k - N_i = \left(\frac{\xi_i + \xi_k}{2} \cos\alpha_{ik} + \frac{\eta_i + \eta_k}{2} \sin\alpha_{ik} \right) s_{ik} \quad . \tag{7}$$

This significantly reduces the number of unknowns, namely, there will be one unknown for each point rather than per side. In an arbitrary network, *there are much less of points than of sides*, since according to the classic principle of triangulation, every new point joins the existing network by two sides. For a homogeneous triangulation network, the side/point ratio may be higher than two. There are another advantages of this solution, e.g. that there is no requirement for writing constraining conditions (6) for the triangles, they being contained in the established observation equations (7). For an interpolation net with m points with known values of geoid heights, with the relevant constraints the number of unknowns may be further reduced, with an additional size reduction of the matrix of normal equations.

Let us see now, how to complete computation for an arbitrary network with more points than needed for an unambiguous solution, where initial geoid heights are known. In this case the unknown N values are determined by adjustment. Relation between components of deflection of the vertical ξ, η and unknown N values of geoid heights is obtained from (7), where

$$C_{ik} = \left(\frac{\xi_i + \xi_k}{2} \cos\alpha_{ik} + \frac{\eta_i + \eta_k}{2} \sin\alpha_{ik} \right) s_{ik} \tag{8}$$

is constant for each triangle side. The question arises what data are to be considered as measurement results for adjustment: the components of deflection of the vertical ξ and η, or C_{ik} values from (8). Since no simple relationship (observation equation) with a

measurement result on one side, and unknowns on the other side of an equation can be written, computation ought to be made under conditions of adjustment of direct measurements, rather than with measured unknowns - this is, however, excessively demanding for computation, requiring excessive storage capacity. Hence concerning measurements, two approximations can be applied: given geoid heights are left uncorrected on the one hand - thus, they are input to adjustment as constraints, C_{ik} on the left hand side of fundamental equation (8) are considered as fictitious measurements and corrected on the other hand. Thereby observation equation (7) becomes:

$$C_{ik} + v_{ik} = N_k - N_i \tag{9}$$

permitting computation under conditions given by adjusting indirect measurements between unknowns.

The first approximation is justified since reliability of given N values exceeds that of the computed values considerably (a principle applied also to geodetic basic networks). Validity of the second approximation will be addressed later in connection with the problem of weighting.

For every triangle side of the interpolated net, observation equation based on Eq. (9):

$$v_{ik} = N_k - N_i - C_{ik} \tag{10}$$

may be written. In matrix form:

$$\underset{(m,1)}{\mathbf{v}} = \underset{(m,n)}{\mathbf{A}} \underset{(n,1)}{\mathbf{x}} + \underset{(m,1)}{\mathbf{l}}$$

where $\mathbf{A}$ is the coefficient matrix of observation equations, $\mathbf{x}$ is the vector containing unknowns N, $\mathbf{l}$ is the vector of constant terms; m is the number of sides in the interpolation net; and n is the number of points. An arbitrary row i of matrix $\mathbf{A}$ is very simple:

$$\begin{bmatrix} 0 & 0 & \dots & 0 & +1 & \dots & -1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

while vector elements of constant term $\mathbf{l}$ are the C_{ik} values.

Adjustment raises also the problem of weighting. Earlier the approximation comprised - rather than direct deflection of the vertical components ξ and η - fictive measurements produced from them. Fictive measurements may only be applied, however, if certain conditions are met. The most important condition is the deducibility of covariance matrix of fictive measurements from the law of error propagation, requiring, however, a relation yielding fictitious measurement results, - in the actual case, *Eq.* (8). Among quantities on the right-hand side of (8), deflection of the vertical components ξ and η may be considered as erroneous. They are about equally reliable $(\pm 0.6'')$, furthermore, they may be considered as mutually independent quantities, thus, their

cofactor matrix $\mathbf{Q}_{\xi\eta}$ will be a unit matrix. With the knowledge of $\mathbf{Q}_{\xi\eta}$, cofactor matrix $\mathbf{Q}_{\mathbf{CC}}$ of fictitious measurements C_{ik} (Detrekői 1991) is:

$$\mathbf{Q}_{\mathbf{CC}} = \mathbf{F}^{*}\mathbf{Q}_{\xi\eta}\mathbf{F} = \mathbf{F}^{*}\mathbf{F}$$

$\mathbf{Q}_{\xi\eta} = \mathbf{E}$ being a unit matrix. Elements of an arbitrary row i of matrix $\mathbf{F}^{*}$ are:

$$\left[\left(\frac{\partial C_{ik}}{\partial \xi}\right)_1 \left(\frac{\partial C_{ik}}{\partial \xi}\right)_2 , \dots , \left(\frac{\partial C_{ik}}{\partial \xi}\right)_n \left(\frac{\partial C_{ik}}{\partial \eta}\right)_1 \left(\frac{\partial C_{ik}}{\partial \eta}\right)_2 , \dots , \left(\frac{\partial C_{ik}}{\partial \eta}\right)_n\right]$$

For the following considerations let us produce rows $\mathbf{f}_1^{*}$ and $\mathbf{f}_2^{*}$ of matrix $\mathbf{F}^{*}$ (referring to sides between points $P_1 - P_2$ and $P_1 - P_3$ respectively):

$$\mathbf{f}_1^{*} = \left[\frac{s_{12}}{2}\left(\cos\alpha_{12}\ , \cos\alpha_{12},\ 0\ , 0\ , \dots , 0\ , \sin\alpha_{12}\ , \sin\alpha_{12}\ , 0\ , 0\ , \dots , 0\ ,\right)\right]$$

and

$$\mathbf{f}_2^{*} = \left[\frac{s_{13}}{2}\left(\cos\alpha_{13}\ , 0\ , \cos\alpha_{13},\ 0\ , \dots , 0\ , \sin\alpha_{13}\ , 0\ , \sin\alpha_{13}\ , 0\ , \dots , 0\ ,\right)\right] .$$

Using $\mathbf{f}_1^{*}$, variance of C value referring to side $P_1 - P_2$ is:

$$m^2 = \frac{s_{12}^2}{4}\left(2\sin^2\alpha_{12} + 2\cos^2\alpha_{12}\right) = \frac{s_{12}^2}{2}$$

while $\mathbf{f}_1^{*}$ and $\mathbf{f}_2^{*}$ yield covariance of C values for sides $P_1 - P_2$ and $P_1 - P_3$:

$$\mathrm{cov} = \frac{s_{12}s_{13}}{4}\left(\sin\alpha_{12}\sin\alpha_{13} + \cos\alpha_{12}\cos\alpha_{13}\right) .$$

Thus, fictitious measurements may be stated to be correlated, and the cofactor matrix contains covariance elements at the junction point of the two sides. If needed, the weighting matrix may be produced by inverting this cofactor matrix. Practically, however, two approximations are possible: either fictitious measurements C are considered to be mutually independent, so weighting matrix is a diagonal matrix; or fictitious measurements are weighted in inverted quadratic relation to the distance.

By assuming independent measurements, the second approximation comes also from inversion, since terms in the main diagonal of the cofactor matrix are proportional to the square of the side lengths. The neglection is, however, justified, in addition to the simplification of computation, also by the fact that contradictions are due less to measurement errors rather than to functional errors of the computational model.

6. Test computations

By a computer program package developed by us we are able to determine deflections of the vertical based on torsion balance measurements and computes geoid heights by astronomical levelling. A characteristic area surrounding *Cegléd* in Hungary measured by torsion balance, extending over some $1200\ km^2$, was chosen for the purpose of test computation. The torsion balance stations were not located with the same density because the observations were carried out with a greater density of points in "disturbed" areas of rugged topography. The interpolation network has 203 points with unknown deflections of the vertical and geoid undulations, and there are 6 points where absolute x, h and N values are known on this area of investigation, referring to *GRS80* system. (3 points are for the initial data of interpolations, and 3 points are for checking of the computations.)

According to our earlier investigations, standard deviations $m_\xi = \pm 0.60''$ and $m_\eta = \pm 0.65''$, computed from the differences at checkpoints corroborate the fact that even for large continuous territories x, h values of acceptable accuracy can be computed from torsion balance measurements (Völgyesi 1995).

Based on our previously interpolated deflection of the vertical components, geoid computations were carried out. The purpose of this test computations is to prove, that the accuracy of Geoid heights computed directly on the arbitrary-shaped network of torsion balance stations is higher than the accuracy of computed values from deflections of the vertical interpolated on a regular square-shaped network points.

It is required to chose an initial point P_1 for geoid computations, where an initial geoid height N_1 is given. The astrogeodetic point *SZOL* was chosen for this purpose at the upper part of our test territory, where $N_1 = 42.74m$, referring to *GRS80* system. Points *13*, *14* and *27* are for checking of computations, geoid heights are given at this points referring to *GRS80* system too. A geoid map can be seen on *Fig. 4* computed from deflections of the vertical interpolated on a regular square-shaped network, while another geoid map can be seen on *Fig. 5* computed directly on the arbitrary-shaped network of torsion balance stations. In *Table I* the given geoid heights of the three check points has been compared to the values computed by the two different methods. The given geoid heights of check points can be found in the 2nd column of *Table I*, geoid heights of *version1* computed from deflections of the vertical interpolated on a regular square-shaped network points can be found in the 3rd column, differences between these and the given values are in the 4th column, geoid heights of *version2* computed directly on the arbitrary-shaped network of torsion balance stations can be found in the 5th column, differences between these and the given values are in the 6th column. It can be stated on the basis of test computations, that the accuracy of geoid heights computed directly on the arbitrary-shaped network of torsion balance stations is higher than the accuracy of computed values from deflections of the vertical interpolated on a regular square-shaped network points - according to our theoretical considerations. The mean error of geoid heights of *version1* is ±0.13m and *version2* is ±0.04m, and the new geoid map contains more tiny details of local geoid forms.

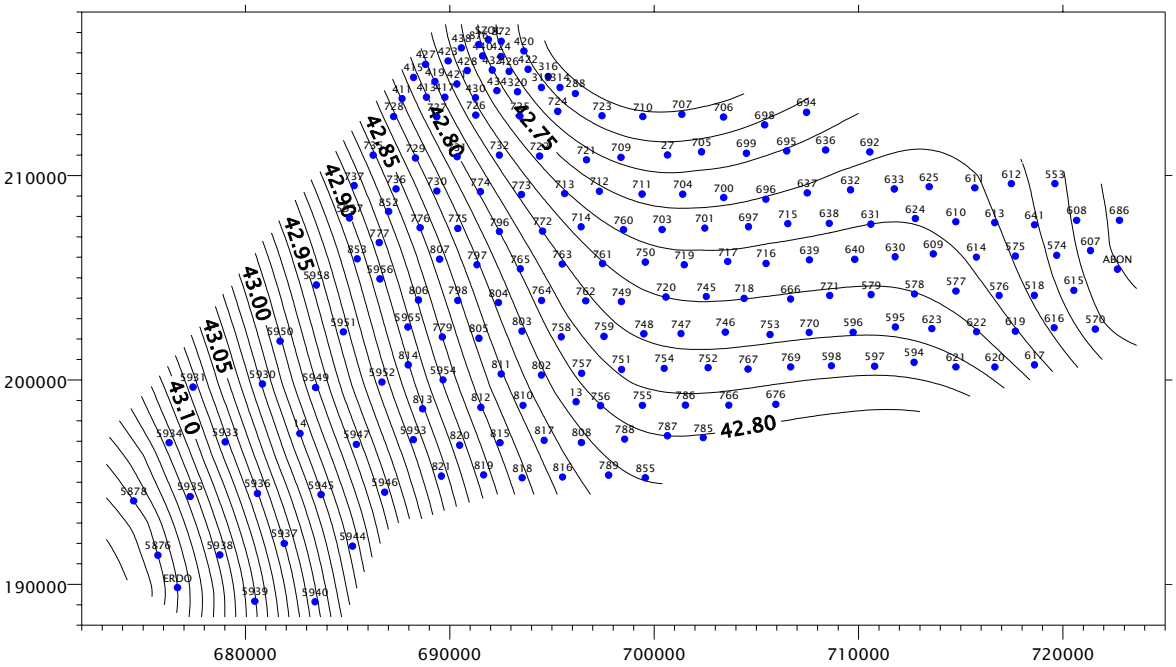

Fig. 4. Geoid map computed from deflections of the vertical interpolated on a regular square-shaped network points

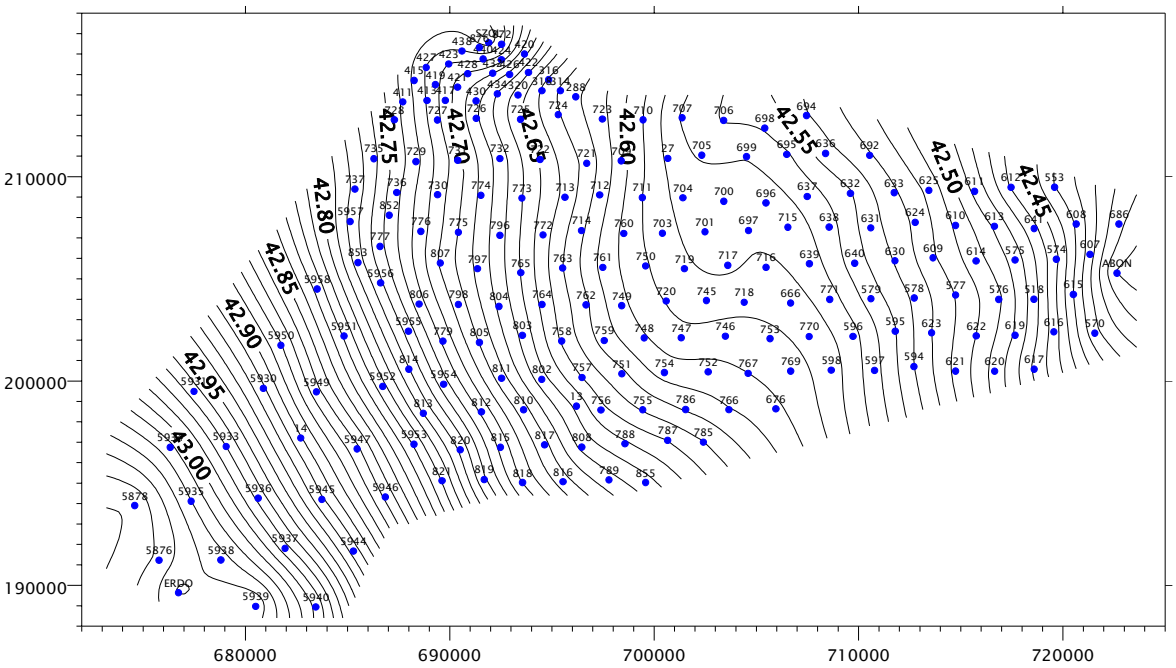

Fig. 5. Geoid map computed directly on the arbitrary-shaped network points of torsion balance stations

***Table I**. Comparison of the two computation method.*

Check points	Given *N* [m]	Computed Version1 [m]	Difference *(V.1-N)* [m]	Computed Version2 [m]	Difference *(V.2-N)* [m]
13	42.72	42.80	0.08	42.66	-0.06
14	42.85	43.00	0.15	42.90	0.05
27	42.58	42.74	0.16	42.58	0.00
			±0.13		±0.04

We are going to make further detailed investigations on other test territories to compare the two computation method.

Acknowledgement

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